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Modular adaptive robust control of SISO nonlinear systems in a semi-strict feedback form

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SUMMARY

In this paper, a modular modification of the adaptive robust control (ARC) technique is presented. The modular design has all of the original ARC properties with an estimation-based update law instead of a Lyapunov-based update law. In this design, the controller is divided into two modules: a control module and an identification module. A key new idea is to set *a priori* bounds on the time derivatives of the estimates to be maintained by the update law. As a result, their effects on the system tracking accuracy can be dominated by the control law. A modification is proposed for the standard gradient and least-square update laws to guarantee the bounds. This modification also makes the controller robust against the generalized (unparameterized) uncertainties considered in the ARC formulation while allowing asymptotic output tracking without the generalized uncertainties. Both the ARC and the modular ARC techniques are applied to a force control problem for an active suspension system. Simulations and experimental results are provided to show that the update law of the modular design is less sensitive to measurement noise which results in smaller force tracking error and smaller control gain. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: adaptive robust control; active suspension; hydraulic actuator

1. INTRODUCTION

The adaptive robust control (ARC) technique was proposed by Yao and Tomizuka [1–4] to combine advantages of adaptive control (AC) and deterministic robust control (DRC) designs while avoiding their drawbacks. For a class of SISO nonlinear systems in a 'semi-strict' feedback form, assuming that bounds of parametric uncertainties are known, the ARC technique guarantees steady-state tracking accuracy and transient tracking accuracy (desirable properties of DRC) and allows asymptotic tracking at the absence of generalized uncertainties using only a continuous control law (a desirable property of AC). The ARC technique has a number of

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successful applications and general procedures for constructing ARC controllers can be found in several papers published in the late 1990s [1–10].

This paper presents a new modular ARC design (MARC) which achieves a controller-identifier separation (modularity) by applying the techniques presented by Krstic *et al.* in adaptive nonlinear control literatures [11, 12]. This allows flexibility in choosing estimation-based gradient and least-square update laws in addition to the original Lyapunov-based update law. Both of the update laws will be presented in this paper. A few important modifications are introduced to maintain the desirable properties of the original ARC design while adding modularity. A key idea is to set *a priori* bounds on the time derivatives of the estimates to be maintained by the update law. As a result, their effects on the system tracking accuracy can be dominated by the control law. These *a priori* bounds can be guaranteed by applying a modification to the parameter update laws similar to the modification used in robust adaptive control techniques [13, 14]. This modification also makes the controller robust against the generalized uncertainties considered in the ARC formulation and doing so without losing asymptotic output tracking if these uncertainties are not present.

The remainder of this paper is organized as follows. In Section 2, a basic formulation of an ARC problem is presented. Then, the control module and the identification module of the modular ARC are described. In Section 3, the ARC and MARC techniques are applied to a force control problem for an active suspension system to illustrate the improved performance achieved by the modular design. Finally, conclusions are made in Section 4.

2. MODULAR ADAPTIVE ROBUST CONTROLLERS

This paper considers an SISO nonlinear system transformable to the following *parametric semi-strict-feedback* form

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \varphi_i(x_1, \ldots, x_i)^{\mathrm{T}}\theta + \Delta_i(t), \quad 1 \leqslant i \leqslant n-1 \\ \dot{x}_n &= u + \varphi_n(x_1, \ldots, x_n)^{\mathrm{T}}\theta + \Delta_n(t) \\ y &= x_1 \end{aligned} \tag{1}$$

where $x_i \in R,\ u \in R,\ y \in R$ are the states, the input and the output of the system, $\Delta_i(t) \in R$ are the generalized uncertainties, $\theta = [\theta_1, \ldots, \theta_p]^{\mathrm{T}} \in R^p$ are unknown parameters and $\varphi_i : R^i \to R^p$ are known smooth functions. θ_i and $\Delta_i(t)$ are assumed to satisfy the following constraints:

$$\theta_i \in \Omega_{\theta_i} \triangleq \{\theta_i : \theta_{im} \leqslant \theta_i \leqslant \theta_{iM}\}, \quad i = 1, \ldots, P \tag{2}$$

$$\Delta_i(t) \in \Omega_{\Delta_i} \triangleq \{\Delta_i : |\Delta_i| \leqslant \Delta_{iM}\}, \quad i = 1, \ldots, n \tag{3}$$

where $\theta_{im},\ \theta_{iM}$ and Δ_{iM} are known constants. In addition, the functions φ_i are assumed to be sufficiently smooth and functions $\Delta_i(t)$ are assumed to be piecewise continuous. The main objectives of the ARC problem are to design a controller such that:

(i) The states of the system (including those of the controller) are bounded.
(ii) The system output tracks a desired output y_r, with adjustable transient output tracking accuracy and steady-state output tracking accuracy.
(iii) The system achieves asymptotic output tracking when $\Delta_k \equiv 0,\ \forall k$.

The desired output, y_r, is assumed to be bounded and has bounded derivatives up to the nth order.

To solve this problem, the original ARC technique [1–4] employs two key ideas: a smooth projection mapping and a dominating control scheme (see Appendices A and B). For each of the estimates of the unknown parameters (θ_i), a projection is used to project it into a bounded set. These projected estimates are used in the control law instead of the estimates so that their effects on tracking accuracy of the system can be arbitrarily minimized by the dominating control scheme. On the other hand, the time derivatives of the estimates, which also affect the tracking accuracy, are cancelled since the adaptation scheme is known.

In this modular design, the controller is separated into a control module and an identification module. Since the control law is to be separated from the update law, the adaptation scheme cannot be used by the control law. However, priori bounds on the time derivatives of the estimates are assumed to be known by the control law and maintained by the update law. As a result, the output tracking performance of the system can be guaranteed (and adjustable) because the effects of the time derivatives of the estimates on tracking performance can be arbitrarily dominated using the same dominating scheme as in the original ARC technique. As long as the bounds are independently maintained by the update law, properties of the control module in the following section are maintained.

2.1. Controller module

The following bounds are assumed by the control module:

$$|\dot{\hat{\theta}}_i| \leqslant M_i, \quad i = 1, \ldots, p \tag{4}$$

where $\hat{\theta}_i$ is an estimate of θ_i and M_i, $i = 1, \ldots, p$ are known constants. Furthermore, a properly designed projection function $\pi_i()$ is assumed for each $\hat{\theta}_i$. In particular, let $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$ and $\hat{\theta}_{i\pi} = \pi_i(\hat{\theta}_i)$ for all $i = 1, \ldots, p$, the projected estimation errors $\tilde{\theta}_{i\pi} \triangleq \theta_i - \pi(\hat{\theta}_i)$ for all $i = 1, \ldots, p$ are bounded; i.e.

$$|\tilde{\theta}_{i\pi}| \leqslant \Pi_i, \quad i = 1, \ldots, p \tag{5}$$

where Π_i, $i = 1, \ldots, p$ are known constants which can be obtained from the projections. Using these two assumptions, the following lemma proposes a control law that guaranteed (ii) and (i) except for boundedness of the states of the identification module.

Lemma 1 (*Controller module design*)

For the system shown in Equation (1), define a positive definite function $V = \frac{1}{2}\sum_{k=1}^{n} z_k^2$, where $z_1 \equiv x_1 - y_r$ and $z_j \equiv x_j - \alpha_{j-1}$ for $1 < j \leqslant n$. The virtual control signal, α_j, are defined as

$$\alpha_j = \begin{cases} -c_1 z_1 - w_1^{\mathrm{T}}\hat{\theta}_\pi + \dot{y}_r - s_1 z_1, & j = 1 \\ -c_j z_j - z_{j-1} + \sum_{k=1}^{j-1} \dfrac{\partial \alpha_{j-1}}{\partial x_k} x_{k+1} + \dfrac{\partial \alpha_{j-1}}{\partial Y^{(j-1)}} \dot{Y}^{(j-1)} - w_j^{\mathrm{T}}\hat{\theta}_\pi - s_j z_j, & 2 \leqslant j \leqslant n \end{cases} \tag{6}$$

where c_j are arbitrary positive constants, $Y^{(k)} \equiv [y_r, \mathrm{d}y_r/\mathrm{d}t, \mathrm{d}^2y_r/\mathrm{d}t^2, \ldots, \mathrm{d}^k y_r/\mathrm{d}t^k]^{\mathrm{T}}$, $w_1 = \varphi_1$, $w_j = \varphi_j - \sum_{k=1}^{j-1} (\partial\alpha_{j-1}/\partial x_k)\varphi_k$ for $2 \leqslant j \leqslant n$, $\hat{\theta}_\pi = [\hat{\theta}_{i\pi}, \ldots, \hat{\theta}_{p\pi}]^{\mathrm{T}}$, and s_j are defined as

$$s_j = \begin{cases} \dfrac{1}{4}\left\{\dfrac{w_1^{2\mathrm{T}}\Pi^2}{\varepsilon_{1,\theta}} + \dfrac{\Delta_{1M}^2}{\varepsilon_{1,1}}\right\}, & j = 1 \\ \dfrac{1}{4}\left\{\dfrac{w_j^{2\mathrm{T}}\Pi^2}{\varepsilon_{j,\theta}} + \left(\dfrac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\right)^2 \dfrac{M^2}{\varepsilon_{j,\dot{\hat{\theta}}}} + \sum_{k=1}^{j-1}\left(\dfrac{\partial\alpha_{j-1}}{\partial x_k}\right)^2 \dfrac{\Delta_{kM}^2}{\varepsilon_{j,k}} + \dfrac{\Delta_{jM}^2}{\varepsilon_{j,j}}\right\}, & 2 \leqslant j \leqslant n \end{cases} \tag{7}$$

The vector $\Pi = [\Pi_1, \ldots, \Pi_p]^{\mathrm{T}}$ contains the estimation error bounds and $M = [M_1, \ldots, M_p]^{\mathrm{T}}$ contains bounds of estimation update rate. $\varepsilon_{j,\theta}$, $\varepsilon_{j,\dot{\hat{\theta}}}$ and $\varepsilon_{i,j}$ are arbitrary positive constants. Square of a vector is defined such that, for example, $M^2 = [M_1^2, \ldots, M_p^2]^{\mathrm{T}}$. Using the control signal $u = \alpha_n$ shown in Equation (6), the followings are obtained:

(1) If we define $\varepsilon_j \equiv p\varepsilon_{j,\theta} + p\varepsilon_{j,\dot{\hat{\theta}}} + \sum_{k=1}^{j} \varepsilon_{j,k}$, then

$$\dot{V} \leqslant -\sum_{k=1}^{n} c_k z_k^2 + \sum_{k=1}^{n} \varepsilon_k$$

which implies that z_i are bounded. It follows that all states x_i are bounded. Transient output tracking accuracy and steady-state output tracking accuracy are guaranteed through the function V and can be arbitrarily adjusted by varying c_k and ε_k [4].

(2) A Z-model of the system (model of the system in the z co-ordinates) is

$$\dot{Z} = A_z Z + W^{\mathrm{T}}\tilde{\theta}_\pi + Q^{\mathrm{T}}\dot{\hat{\theta}} + R^{\mathrm{T}}\Delta \tag{8}$$

where $Z \equiv [z_1, \ldots, z_n]^{\mathrm{T}}$

$$A_z = \begin{bmatrix} -c_1 - s_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 - s_2 & 1 & \cdots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -c_{n-1} - s_{n-1} & 1 \\ 0 & \cdots & \cdots & -1 & -c_n - s_n \end{bmatrix}, \quad W^{\mathrm{T}} = \begin{bmatrix} w_1^{\mathrm{T}} \\ \vdots \\ w_i^{\mathrm{T}} \\ \vdots \\ w_n^{\mathrm{T}} \end{bmatrix},$$

$$\tilde{\theta}_\pi = \begin{bmatrix} \tilde{\theta}_{1\pi} \\ \vdots \\ \tilde{\theta}_{p\pi} \end{bmatrix},$$

$$Q^{\mathrm{T}} = \begin{bmatrix} \underline{0} \\ \dfrac{\partial \alpha_1}{\partial \hat{\theta}} \\ \vdots \\ \dfrac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \end{bmatrix}$$

$$\hat{\theta} = \begin{bmatrix} \hat{\theta}_1 \\ \vdots \\ \hat{\theta}_p \end{bmatrix}, \quad R^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ -\dfrac{\partial \alpha_1}{\partial x_1} & 1 & 0 & \dots & \vdots \\ -\dfrac{\partial \alpha_2}{\partial x_1} & -\dfrac{\partial \alpha_2}{\partial x_2} & 1 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ -\dfrac{\partial \alpha_{n-1}}{\partial x_1} & \dots & \dots & -\dfrac{\partial \alpha_{n-1}}{\partial x_{n-1}} & 1 \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} \Delta_1 \\ \vdots \\ \Delta_n \end{bmatrix}$$

Proof
see Appendix C. □

2.2. Identification module

The identification module is designed in two steps. First, a swapping model is constructed to simplify the identification process by transforming the dynamic-system identification problem into a static problem. The second step is to design update laws based on the swapping model. Two update laws are presented in this paper: a gradient and a least-square update law. The update laws are modified to obtain boundedness of the estimates and boundedness of the time derivatives of the estimates (Equation (4)). Boundedness of the estimates is required for stability of the identification module but does not affect the control module since the projected estimates are used in the control law. Except for the added complexity resulting from the modification and the smooth projection used, the following proof closely follows [12].

Lemma 2 (*Z-swapping scheme*)
Assuming the results from Lemma 1, consider the Z-model shown in Equation (8):

$$\dot{Z} = A_z Z + W^{\mathrm{T}} \tilde{\theta}_\pi + Q^{\mathrm{T}} \dot{\hat{\theta}} + R^{\mathrm{T}} \Delta \tag{9}$$

and the following filters:

$$\dot{\Omega}^{\mathrm{T}} = A_z \Omega^{\mathrm{T}} + W^{\mathrm{T}}, \quad \Omega^{\mathrm{T}} \in R^{n \times p}, \quad \Omega(t_0) = 0 \tag{10}$$

$$\dot{\Omega}_0^{\mathrm{T}} = A_z \Omega_0^{\mathrm{T}} + W^{\mathrm{T}} \hat{\theta}_\pi - Q^{\mathrm{T}} \dot{\hat{\theta}}, \quad \Omega_0 \in R^n, \quad \Omega_0(t_0) = -Z(t_0) \tag{11}$$

If it is assumed further that the identification law is designed such that $\hat{\theta}$ is bounded, it can be shown that:

(1) Z is related to the augmented states Ω and Ω_0 in the following form:

$$Z = \Omega^{\mathrm{T}}\theta - \Omega_0^{\mathrm{T}} + \tilde{e} \tag{12}$$

where

$$\dot{\tilde{e}} \equiv A_z\tilde{e} + R^{\mathrm{T}}\Delta \quad \text{and} \quad \tilde{e}(t_0) = 0 \tag{13}$$

(2) If the generalized uncertainty $\Delta \neq 0$, we have that $\tilde{e}, \Omega$ and Ω_0 are bounded. If $\Delta \equiv 0$ (or becomes zero after a finite time), we have that $\tilde{e}$ converges to zero exponentially. As a result, $\tilde{e} \in L_2$.

(3) Let an estimate of Z be $\hat{Z} = \Omega^{\mathrm{T}}\hat{\theta} - \Omega_0^{\mathrm{T}}$, then the estimation error ($e \equiv Z - \hat{Z}$) is related to the parameter error $\tilde{\theta} = \theta - \hat{\theta}$ by

$$e = Z - \hat{Z} = \Omega^{\mathrm{T}}\tilde{\theta} + \tilde{e} \tag{14}$$

Furthermore, assuming that $\hat{\theta}$ is bounded, e is bounded.

Proof

(1) Differentiating Equation (12), we get

$$A_zZ + W^{\mathrm{T}}\tilde{\theta}_\pi + Q^{\mathrm{T}}\dot{\hat{\theta}} + R^{\mathrm{T}}\Delta = A_z\Omega^{\mathrm{T}}\theta + W^{\mathrm{T}}\theta - A_z\Omega_0^{\mathrm{T}} - W^{\mathrm{T}}\hat{\theta}_\pi + Q^{\mathrm{T}}\dot{\hat{\theta}} + \dot{\tilde{e}}$$

Note that $\tilde{\theta}_\pi = \theta - \hat{\theta}_\pi$, the above equation becomes $\dot{\tilde{e}} = A_z(Z - \Omega^{\mathrm{T}}\theta + \Omega_0^{\mathrm{T}}) + R^{\mathrm{T}}\Delta$. Using Equation (12), we obtain Equation (13).

(2) Let $R^{\mathrm{T}}\Delta = [r_1, \dots r_n]^{\mathrm{T}}$ and a positive definite function $V = \frac{1}{2}\sum_{k=1}^{n} \tilde{e}_k^2$, it may be shown that

$$\dot{V} \leqslant \sum_{k=1}^{n} (-c_0\tilde{e}_k^2 + r_k\tilde{e}_k) \tag{15}$$

where $c_0 = \min_i(c_i)$. Equation (15) implies that $\tilde{e}$ is bounded whenever R is bounded. Since R is a continuous vector function of X (and is defined for any value of X), they are bounded if X is bounded. Similarly, Ω_0 is also bounded since W is also a continuous function of X. For boundedness of Ω, we consider a positive definite function $V = 1/2|\Omega|_{\mathrm{F}}^2 = 1/2\,\mathrm{tr}\{\Omega^{\mathrm{T}}\Omega\}$, where $|\,|_{\mathrm{F}}$ is the Frobenius norm. It can be shown that

$$\begin{aligned}\dot{V} &= 1/2\,\mathrm{tr}\{\Omega(A_Z + A_Z^{\mathrm{T}})\Omega^{\mathrm{T}} + \Omega^{\mathrm{T}}W + W^{\mathrm{T}}\Omega\}\\ &\leqslant -c_0\,\mathrm{tr}\{\Omega\Omega^{\mathrm{T}}\} + \mathrm{tr}\{W^{\mathrm{T}}\Omega\}\\ &\leqslant -\frac{c_0}{2}|\Omega|_{\mathrm{F}}^2 - \sum_{i=1}^{n}\left\{-\frac{c_0}{2}|\Omega_i|^2 + w_i^{\mathrm{T}}\Omega_i\right\}\end{aligned}$$

where Ω_i is the ith column vector of Ω.

By completing the square we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}|\Omega|_{\mathrm{F}}^2\right) \leqslant -\frac{c_0}{2}|\Omega|_{\mathrm{F}}^2 + \frac{|w_i|^2}{2c_0}$$

which implies boundedness of all elements of Ω because W is bounded. When $\Delta \equiv 0$, Equation (15) becomes $\dot{V} \leqslant -\sum_{k=1}^{n} c_0 \tilde{e}_k^2$ or $V \leqslant V(t_0)e^{-2c_0 t}$; i.e. $\tilde{e}$ converges to zero exponentially.

(3) Equation (14) is obtained directly by using Equation (12). Boundedness of e follows directly from Equation (14) since Ω, $\hat{\theta}$ and $\tilde{e}$ are all bounded. □

2.2.1. Parameter update laws

The parameter update laws used in this paper have the form

$$\dot{\hat{\theta}} = \Gamma\{\mu(\hat{\theta}) + \Theta(\Omega e)\}, \quad \hat{\theta}_i(t_0) \in \Omega_{\theta_i} \quad \forall i = 1, \ldots, p \tag{16}$$

where Γ is a matrix depending whether a gradient or a least-square update law is used, $\mu(\hat{\theta}) = [\mu_1(\hat{\theta}_1), \ldots, \mu_p(\hat{\theta}_p)]^{\mathrm{T}}$,

$$\mu_i(\hat{\theta}_i) = \begin{cases} (\theta_{im} - \hat{\theta}_i), & \hat{\theta}_i < \theta_{im} \\ 0, & \theta_{im} \leqslant \hat{\theta}_i \leqslant \theta_{iM} \\ -(\hat{\theta}_i - \theta_{iM}), & \hat{\theta}_i > \theta_{iM} \end{cases} \tag{17}$$

$$\Theta(\Omega e) = \begin{cases} \Omega e, & |\Omega e| \leqslant M' \\ M' \dfrac{\Omega e}{|\Omega e|}, & |\Omega e| > M' \end{cases} \tag{18}$$

and M' is an adjustable constant. For a gradient update law, Γ is a positive definite constant matrix. For a least-square update law, Γ is a covariance matrix where

$$\dot{\Gamma} = -\Gamma^{\mathrm{T}}\Omega\Omega^{\mathrm{T}}\Gamma\phi \tag{19}$$

where $\Gamma(t_0) \in R^{p\times p}$ is a positive definite matrix, and

$$\phi = \begin{cases} M'/|\Omega e| & \text{when } |\Omega e| > M' \\ 1 & \text{otherwise} \end{cases}$$

Furthermore, a covariance resetting is also used with

$$\Gamma(t_r^+) = \rho_r I$$

where ρ_r is a positive constant and t_r is the time for which $\underline{\lambda}(\Gamma)/\bar{\lambda}(\Gamma) \leqslant r < 1$, for a positive constant r. Note that this covariance resetting still allows Γ to be arbitrarily small.

Without the modification functions $\mu(\,)$, $\Theta(\,)$ and ϕ, the update laws become standard update laws based on Equation (14) when $\tilde{e} = 0$ ($\Delta \equiv 0$). In fact, when the general uncertainty does not exist ($\Delta \equiv 0$), asymptotic output tracking can be proven. However, we will first establish boundedness of $\hat{\theta}_i$ and $\dot{\hat{\theta}}_i$, $\forall i$, whether or not $\Delta \equiv 0$. This is guaranteed because of the functions $\mu(\,)$ and $\Theta(\,)$. To understand how they work, consider a case where a gradient update law is used with a diagonal matrix Γ. For each θ_i, we have $\dot{\hat{\theta}}_i = \gamma_i\{\mu_i(\hat{\theta}_i) + \Theta_i(\Omega e)\}$ where γ_i is the i, i component of the matrix Γ and $\Theta_i(\Omega e)$ is the ith component of the vector $\Theta(\Omega e)$. Next, consider a scalar system, $\dot{x} = a(-x + c)$, where a is a positive constant and $|c(t)| < c_M$. It is clear that x in this system is bounded because c is bounded. It is also clear that $\dot{x}$ is bounded since x and c are bounded. In fact the bound on $\dot{x}$ can be made arbitrarily small if c_M can be adjusted arbitrarily small. The function $\mu_i()$ works similarly to the function $-x$ but by attracting the estimate $\hat{\theta}_i$, toward the set $\Omega_{\theta i}$ rather than to zero. Furthermore, because the magnitude of $\Theta_i(\Omega e)$ cannot be larger than M', it can be easily shown that both $\hat{\theta}_i$, and $\dot{\hat{\theta}}_i$ are bounded. In fact, $\dot{\hat{\theta}}_i$ is bounded by $2\gamma_i M'$ which can be arbitrarily adjusted to satisfy Equation (4).

When Γ is not diagonal, we will consider the more complicated case of a least-square update law where Γ may not be diagonal and can be time varying. Let $V = \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta}$ and $h_i = \tilde{\theta}_i - \mu_i(\hat{\theta}_i)$, then we have

$$\dot{V} = -\tilde{\theta}^{\mathrm{T}}\Gamma\{\tilde{\theta} - h + \Theta(\Omega e)\} \leqslant -\underline{\lambda}(\Gamma)|\tilde{\theta}|^2 + |\tilde{\theta}|\bar{\lambda}(\Gamma)\{|h| + M'\}$$

$$\leqslant \bar{\lambda}(\Gamma)|\tilde{\theta}|\{-|\tilde{\theta}|\underline{\lambda}(\Gamma)/\bar{\lambda}(\Gamma) + |h| + M'\}$$

where $h = [h_1, \ldots, h_p]^{\mathrm{T}}$. Since $\underline{\lambda}(\Gamma)/\bar{\lambda}(\Gamma)$ is bounded from below by r, we have $\dot{V} \leqslant r\bar{\lambda}(\Gamma)|\tilde{\theta}|\{-|\tilde{\theta}| + (|h| + M')/r\}$. Since $|h_i| \leqslant \theta_{iM} - \theta_{im}$ (see Figure 1), it follows that $\dot{V} \leqslant 0$ whenever $\tilde{\theta}$ is outside the set

$$S \triangleq \{\tilde{\theta} : |\tilde{\theta}| \leqslant \frac{1}{r}\left\{\left(\sum_i (\theta_{iM} - \theta_{im})^2\right)^{1/2} + M'\right\}\}$$

Because, $\tilde{\theta}$ starts inside S, we have that $\tilde{\theta}(t) \in S$, $\forall t \geqslant 0$; i.e. $\hat{\theta}$ is bounded. Since $|\tilde{\theta}_i| \geqslant |\mu_i(\hat{\theta}_i)|$, it follows from Equation (16) that

$$|\dot{\hat{\theta}}| \leqslant \rho_0\left(\frac{1}{r}\left\{\left(\sum_i (\theta_{iM} - \theta_{im})^2\right)^{1/2} + M'\right\} + M'\right)$$

where $\rho_0 = \max(\rho_r, \bar{\lambda}(\Gamma(t_0)))$. This bound can be set arbitrarily small by adjusting ρ_0, r and M'. Properties of the update laws are summarized below.

Fact 1: The estimates $\hat{\theta}_i, \forall i$ are bounded.

Fact 2: $\dot{\hat{\theta}}_i, \forall i$ are bounded and the bounds can be adjusted arbitrarily to satisfy Equation (4).

Fact 3: $\tilde{\theta}^{\mathrm{T}}\mu(\hat{\theta}) \geqslant \mu(\hat{\theta})^{\mathrm{T}}\mu(\hat{\theta}), \forall \theta_i \in [\theta_{im}, \theta_{iM}]$. This is true since (i) $|\theta_i - \hat{\theta}_i| \geqslant |\mu_i(\hat{\theta}_i)|$, and (ii) $(\theta_i - \hat{\theta}_i)$ and $\mu_i(\hat{\theta}_i)$ always have the same sign whenever $\mu_i(\hat{\theta}_i) \neq 0$ (see Figure 1).

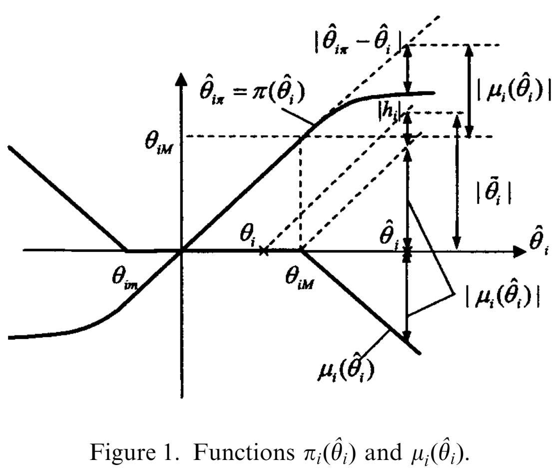

Figure 1. Functions $\pi_i(\hat{\theta}_i)$ and $\mu_i(\hat{\theta}_i)$.

Fact 4: $\mu(\hat{\theta}) \in L_2$ implies that $(\hat{\theta}_\pi - \hat{\theta}) \in L_2$. This fact is a direct result of $|\mu_i(\hat{\theta}_i)| \geqslant |\hat{\theta}_{i\pi} - \hat{\theta}_i|$, which can be seen in Figure 1.

Fact 5: For the least-square update law, $\Gamma(t)$ is symmetric, positive definite and Γ^{-1} is non-increasing for any x. Hence, Γ^{-1} is positive definite and $x^{\mathrm{T}}\Gamma x$ is non-decreasing for any x.

It is important to emphasize that these facts are independent of how Ω or e is defined. As a result, they do not get affected by the control law, the generalize uncertainty Δ or the filters in Lemma 2. They can be used to establish boundedness of the system states using Lemma 1 and the filter systems using Lemma 2. Next, we present properties of the update laws when $\Delta \equiv 0$ which will be used to show asymptotic output tracking of the system.

Lemma 3 (*Properties of the Update Laws when* $\Delta \equiv 0$)
Given the systems described in Equations (9)–(11), and the update laws of Equation (16), we have (1) $e \in L_2$, (2) $\mu(\hat{\theta}) \in L_2$, (3) $\dot{\hat{\theta}} \in L_2$, (4) $\Omega^{\mathrm{T}}\tilde{\theta} \in L_2$.

Proof
See Appendix D. It will be sufficient here to note that Facts 1 and 2 imply the results from Lemmas 1 and 2 which are used in this proof along with Fact 3. Note that the proof can be easily applied when $\Delta = 0$ after some finite time as well. □

Theorem 1 (*Modular ARC*)
Given a system in the form of Equation (1), a modular ARC controller can be constructed using the controller module (Lemma 1 ($u = \alpha_n$)), and the identification module (Equations (10), (11) and (16)). The resulting Modular ARC controller has the following properties:

(1) The system states and control signals are bounded.
(2) The system output has adjustable transient tracking accuracy and steady-state tracking accuracy.
(3) The system output tracks the desired output (y_r) asymptotically when $\Delta \equiv 0$.

Proof

(1) Boundedness of $X, \hat{\theta}_i, \Omega$ and Ω_0 was shown in Fact 1, Lemmas 1 and 3. The control signals are bounded since the control law is continuous and its parameters are bounded.
(2) This follows directly from Fact 2 and Lemma 1.
(3) This statement is equivalent to $Z \to 0$. Using Barbalat's Lemma, it is adequate to show that $\dot{Z}$ is bounded and $Z \in L_2$. Boundedness of $\dot{Z}$ can be deduced from Equation (9), since $\hat{\theta}$ is bounded (Fact 2), $\tilde{\theta}_\pi$ is bounded by definition, Z and X are bounded (from Lemma 1), and the matrices A_z, W^{T}, and Q^{T} are bounded (because they are continuous functions of Z and X). Let
$$\psi^{\mathrm{T}} = \Omega_0^{\mathrm{T}} - \Omega^{\mathrm{T}}\hat{\theta} \tag{20}$$
Using Equations (12) and (20), we have
$$Z = \Omega^{\mathrm{T}}\theta - (\Omega_0^{\mathrm{T}} - \Omega^{\mathrm{T}}\hat{\theta}) - \Omega^{\mathrm{T}}\hat{\theta} + \tilde{e} = \Omega^{\mathrm{T}}\tilde{\theta} - \psi + \tilde{e} \tag{21}$$

From Equation (21), it is clear that since $\tilde{e} \in L_2$ (Lemma 2(2)) and $\Omega^{\mathrm{T}}\tilde{\theta} \in L_2$ (Lemma 3), $\psi \in L_2$ implies that $Z \in L_2$. Therefore, it is only needed to show that $\psi \in L_2$. Differentiating Equation (20), we have

$$\dot{\psi}^{\mathrm{T}} = A_z\psi^{\mathrm{T}} - \Omega^{\mathrm{T}}\dot{\hat{\theta}} - Q^{\mathrm{T}}\dot{\hat{\theta}} + W^{\mathrm{T}}(\hat{\theta}_\pi - \hat{\theta}) \tag{22}$$

Let $T = -Q^{\mathrm{T}}\dot{\hat{\theta}} + W^{\mathrm{T}}(\hat{\theta}_\pi - \hat{\theta})$, Equation (22) can be rewritten as

$$\dot{\psi}^{\mathrm{T}} = A_z\psi^{\mathrm{T}} + T \tag{23}$$

Using Fact 4, and Lemma 3 ($\mu(\hat{\theta}) \in L_2$), we have $(\hat{\theta}_\pi - \hat{\theta}) \in L_2$. Since $\dot{\hat{\theta}} \in L_2$ (Lemma 3) and both Q and W are bounded, $T \in L_2$. With $T = 0$, let $V = \sum_{k=1}^{n} \frac{1}{2}\psi_k^2$, similar to the proof of Lemma 2, we can see that $V \leqslant V(t_0)e^{-2c_0 t}$. Therefore, the state transition matrix of Equation (23) satisfies $|\Phi(t,\tau)| \leqslant ke^{-\alpha(t-\tau)}$, for some positive k and α. Hence, the solution of Equation (23) satisfies

$$|\psi(t)| \leqslant ke^{-\alpha t}|\psi(0)| + k\int_0^t e^{-\alpha(t-\tau)}|T(\tau)|\,\mathrm{d}\tau$$

and we have

$$\int_0^t |\psi(t)|^2\,\mathrm{d}t \leqslant \frac{k^2}{\alpha}|\psi(0)|^2 + 2\frac{k^2}{\alpha^2}\int_0^t |T(s)|^2\,\mathrm{d}s$$

Since $T \in L_2$, we have $\psi \in L_2$. □

3. SIMULATION AND EXPERIMENTAL RESULTS

In this section, the ARC and MARC techniques are applied to a force control problem for an active suspension system [10, 15]. For the force control problem, the objective of the controller is to make the actual force follow the desired force generated by an outer-loop controller closely. Numerous papers have been published on the outer-loop designs but most of them did not consider the force dynamics and assumed that accurate force can be generated as desired. Since designing a good force controller is not trivial, a number of papers have been published on the force control problem (e.g. References [16–19]). A solution is to use the ARC controller [10]. However, it is found from experience by the authors that the update law of the ARC controller usually gives diverging estimates in this force tracking application. As a result, the controller becomes reliant on high gains to provide good tracking. A high gain controller may not be desirable in active suspension applications where quality of the sensors and computing power may be limited and where high-frequency modes of the hydraulic actuator are usually neglected. As a result, the MARC technique was applied which resulted in improved estimation.

The main objective of this section is to provide a comparison between an ARC and an MARC controller in terms of tracking accuracy and identification accuracy. It is important to note that these results are not intended for proving that the modular design is better than the original design. However, it is our purpose here to show that, modular design can be beneficial in certain applications. For a hydraulic actuator, the dynamics of the force sub-loop is

$$\dot{F}_{\mathrm{a}} = \theta_1[k_1(\dot{x}_w - \dot{x}_c) - k_2F_{\mathrm{a}} + k_3u] + d$$

where F_a is the actuator force (N), θ_1 is an unknown parametric uncertainty (which arises from fluid bulk modulus, pipe flexibility, etc. and could change significantly but slowly), k_1, k_2 and k_3 are known vehicle/actuator parameters, $\dot{x}_w - \dot{x}_c$ is the rate of change of suspension stroke which is treated as a measured disturbance, u is the virtual control input resulting from a feedback linearization scheme, and d is the general uncertainty. It is important to note that the equation is in a form slightly different from Equation (1). However, the procedure is only needed to be modified slightly for this simple system if we also assumed that the unknown parameter is positive, i.e. $0 < \theta_{1m} < \theta_1 < \theta_{1M}$. Finally, for simplicity, d will not be considered. It may be included to check if the controller's states will be bounded as claimed but this is not our intention.

3.1. ARC controller

The original ARC force controller has the following form:

$$u = u_{1a} + u_{1s} \tag{24}$$

$$\dot{\hat{\theta}}_1 = \gamma_1 \tau_1 \tag{25}$$

where $\tau_1 = z_1(k_1(\dot{x}_w - \dot{x}_c) - k_2 F_a + k_3 u_{1a})$, γ_1 is an arbitrary positive gain. The adaptive and robust control signals are computed from

$$u_{1a} = \frac{1}{k_3}\left\{-k_1(\dot{x}_w - \dot{x}_c) + k_2 F_a + \frac{1}{\hat{\theta}_{1\pi}}(\dot{F}_d - c_1 z_1)\right\} \tag{26}$$

and $u_{1s} = -s_1 z_1$, where

$$s_1 = \frac{1}{4\theta_{1m}k_3}\left(\frac{1}{\varepsilon_{11}}\Pi_1^2(k_1(\dot{x}_w - \dot{x}_c) - k_2 F_a + k_3 u_{1a})^2 + \frac{1}{\varepsilon_{12}} d_M^2\right)$$

c_1, ε_{11} and ε_{12} are positive control gains, F_d is the desired force and $z_1 \equiv F_a - F_d$. Defining $V_1 = \frac{1}{2} z_1^2$, it can be shown that

$$V_1(t) \leqslant e^{-2c_1 t} V_1(t_0) + \frac{\varepsilon_{11} + \varepsilon_{12}}{2c_1} \tag{27}$$

which is used to guarantee tracking performance according to (ii). Asymptotic tracking when $d = 0$ can also be proven by showing that

$$\dot{V}_2 \leqslant -c_1 z_1^2 \quad \text{for } V_2 = \frac{1}{2} z_1^2 + \frac{1}{\gamma_1}\int_0^{\tilde{\theta}_1} (\theta_1 - \pi(\theta_1 - v))\, \mathrm{d}v$$

For details, please refer to Reference [10].

The ARC controller needs the signal $\dot{F}_d$ in its control law. As an alternative to differentiating $\dot{F}_d$, the following command signal filter is used:

$$\dot{v} = -a_f v + F_d$$

$$Y = \begin{bmatrix} 1 \\ -a_f \end{bmatrix} v + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F_d$$

where a_f is positive constant, $v \in R$, $Y = [F_{df}, \dot{F}_{df}]^T$, and F_{df} is the filtered desired force signal. F_{df} and its derivative, $\dot{F}_{df}$, are used in the ARC controller instead of F_d and $\dot{F}_d$. This will also be used in the MARC controller.

3.2. Modular ARC controller

Note that for a first-order system, the adaptation speed does not show up in the design. Since ARC and MARC differ in how they treat the adaptation speed, their control laws are exactly the same. As a result, this problem is perfect for comparing their update laws because any performance difference can be attributed directly to the update laws. For the identification module, we chose to use the least-square update law since it is in general more robust. The Z-dynamics for this example problem is

$$\dot{z}_1 = A_z z_1 + \tilde{\theta}_{1\pi} w_1 + d$$

where $A_z = -c_1 - \hat{\theta}_{1\pi} k_3 s_1$ and $w_1 = k_1(\dot{x}_w - \dot{x}_c) - k_2 F_a + k_3 u_{1a} - k_3 s_1 z_1$. The least-square identification law is $\dot{\hat{\theta}}_1 = \Gamma\{\mu(\hat{\theta}_1) + \Theta(\Omega e_1)\}$, where $e_1 \equiv z_1 - \Omega\hat{\theta}_1 + \Omega_0$, $\dot{\Gamma} = -\Gamma^2\Omega^2\phi$, $\dot{\Omega} = A_z\Omega + w_1$ and $\dot{\Omega}_0 = A_z\Omega_0 + w_1\hat{\theta}_{1\pi}$.

In the following simulations, the desired force signal $F_d = 200\sin(0.5 \times 2\pi t) + 100\sin(3.1 \times 2\pi t) + 50\sin(1.1 \times 2\pi t)$. The parameter values used are $c_1 = 140, \varepsilon_{11} = \varepsilon_{12} = 60000, d_M = 0$, $\theta_{1m} = 0.6\theta_1, \theta_{1M} = 1.4\theta_1, \hat{\theta}_1(t_0) = 0.8\theta_1, \gamma_1 = 5 \times 10^{-5}$ and $a_f = 62$ (10 Hz command signal filter bandwidth). The plant parameters are $k_1 = 2.6 \times 10^{-6}, k_2 = 0, k_3 = 1.75 \times 10^{-6}$ and $\theta_1 = 3.8 \times 10^{11}$. The signal $\dot{x}_w - \dot{x}_c$ is generated from a suspension model using a transfer function from F_a to $\dot{x}_w - \dot{x}_c$ as described in Reference [10]. A sampling rate of 2000 Hz is used for all of the measurement and input signals. The measurement noise was implemented using a band-limited white noise generator with its standard deviation matched to that of the corresponding signal measured from the University of Michigan Active Suspension Test Rig [15]. For the extra parameters of the MARC controller, we use $\Gamma(0) = 1$ while functions $\mu(\ )$ and $\Theta(\ )$ are chosen such that they will not become active. Furthermore, the covariance resetting is not required since $\underline{\lambda}(\Gamma(t))/\bar{\lambda}(\Gamma(t)) = 1, \forall t$.

The simulation results are shown in Figures 2 and 3. Figure 2 shows tracking errors between the actual force and the desired force (F_{df}). In terms of RMS values (root mean square value), force error of MARC controller is about 50% less (5.45 for ARC and 2.71 for MARC). Figure 3 shows the difference between the virtual control signals $u(t)$ and their ideal values (u_{1a} when the estimate is correct). This plot can be considered to show unnecessary input signals generated by the controller. Since the control laws of the two controllers are identical, larger input signal and larger force error of the ARC controller can be traced back to the term $(\dot{F}_d - c_1 z_1)/\hat{\theta}_{1\pi}$ and its inaccurate estimate (to be shown in the next paragraph).

Trajectories of the estimates are shown in Figure 4 where measurement noises are generated as described above with their standard deviations set at 100, 50, 25 and 1% of the measured value. Similarly, Figure 5 shows effects of the noises on the estimation accuracy of the controllers. It can be seen that the estimates from the ARC controller will not converge when the noise levels are larger 50% (the estimates are allowed to be lower than $0.6\theta_1$ in this plot). This figure shows that the estimation accuracy of the MARC controller is much less sensitive to measurement noise.

Estimation accuracy is verified experimentally using the University of Michigan Active Suspension Test Rig under a realistic road excitation and an LQ outer-loop controller (for

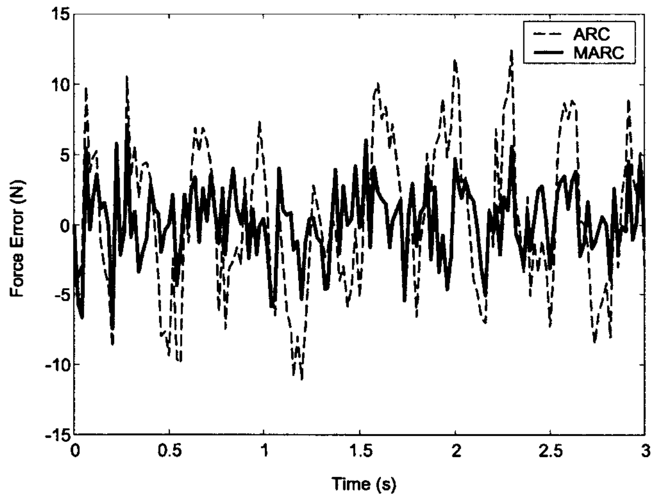

Figure 2. Force tracking simulations.

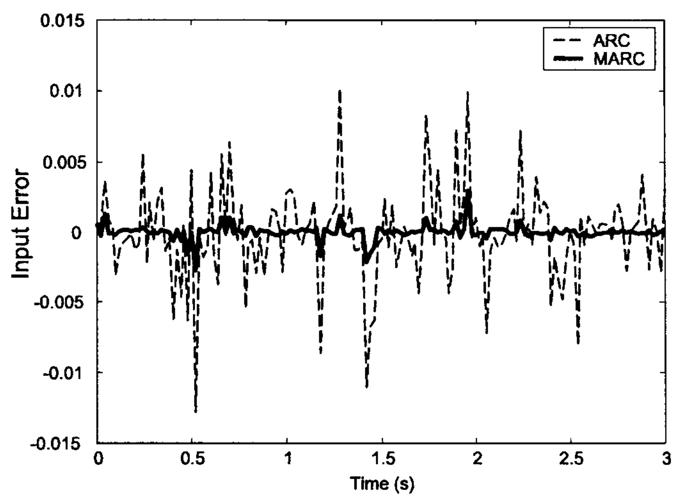

Figure 3. Control signals of ARC and MARC algorithms.

calculating the desired force). For details on description of the test rig, the outer-loop controller, the road excitation and the parameters used please refer to References [10, 15]. Figure 6 shows trajectory of the estimate of the ARC controller. As predicted by the simulation, the estimate diverges to the lower bound, which makes the control law use higher gain according to the term $(\dot{F}_{\mathrm{d}} - c_1 z_1)/\hat{\theta}_{1\pi}$. In fact, high gain instability may occur if the lower bound θ_{1m} is too small as shown by the actual control signal i_{sv} (roughly proportional to the virtual signal u) after around 28 s.

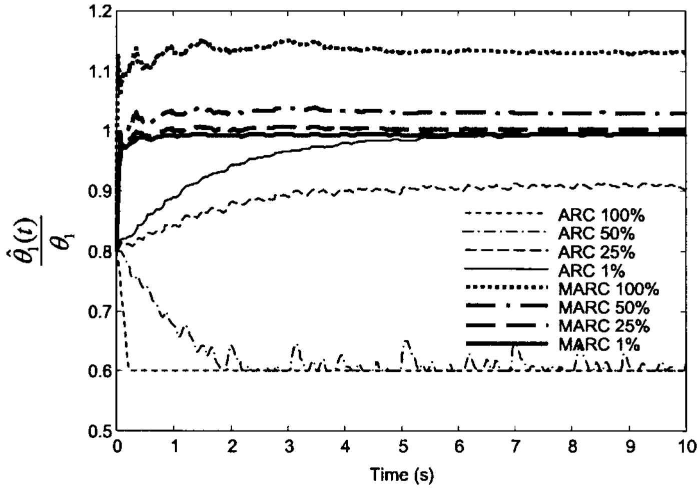

Figure 4. $\hat{\theta}_1$ estimation trajectory.

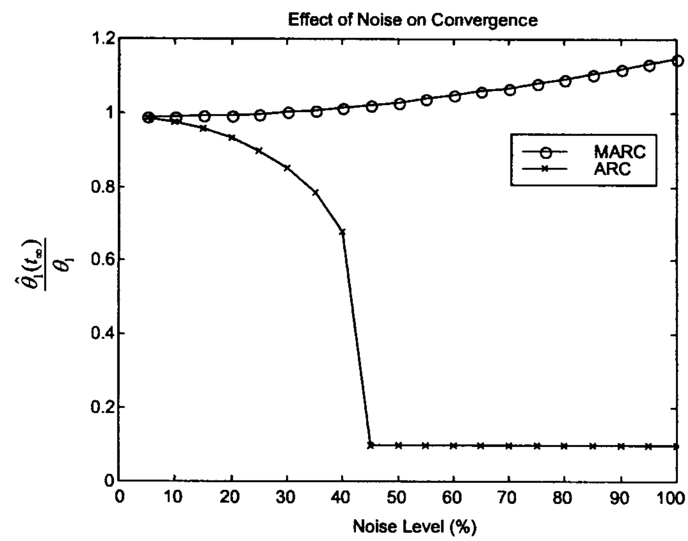

Figure 5. Effects of noise level on parameter estimation.

For the MARC controller, it is found that the estimate is not as accurate as predicted by the simulations, possibly due to friction in the actuator. However, being an estimation-based identification, the update law can be easily modified by noting that the measured force $F_m = F_a + k_f(\dot{x}_w - \dot{x}_c)$, where F_a is the force resulting from hydraulic pressures only and the friction is modelled simply as a damper with an unknown constant k_f. Hence,

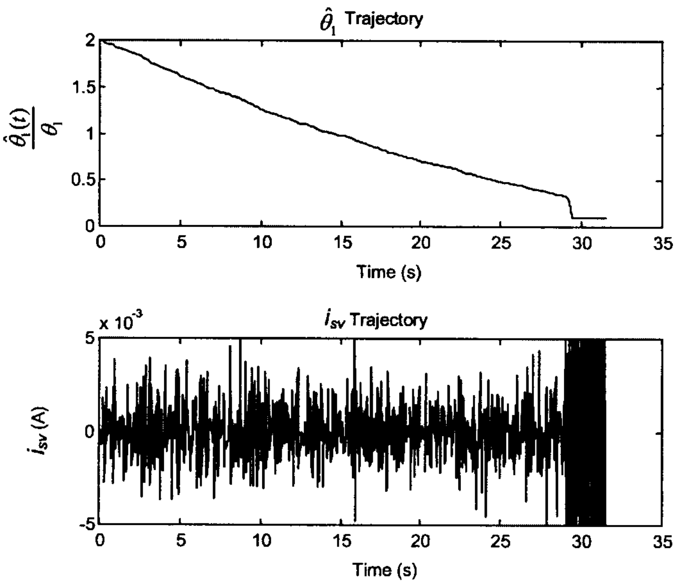

Figure 6. Experimental result of LQ-ARC ($\hat{\theta}_1(t_0) = 2\theta_1$).

we have

$$e_1 = \Omega^{\mathrm{T}}\tilde{\theta}_1 + \tilde{k}_{\mathrm{f}}(\dot{x}_w - \dot{x}_c) + \tilde{e}_1 \tag{28}$$

where e_1 is calculated using the measured force; i.e. $e_1 = F_{\mathrm{m}} - F_{\mathrm{d}} - \Omega\hat{\theta}_1 + \Omega_0$, $\tilde{k}_{\mathrm{f}} = k_{\mathrm{f}} - \hat{k}_{\mathrm{f}}$ and $\hat{k}_{\mathrm{f}}$ is an estimate of k_{f}. Using Equation (28), the update law is modified as

$$\begin{aligned}
\dot{\hat{\theta}}_1 &= \gamma_1(P_1 + p_1)(\mu(\hat{\theta}_1) + \Theta(\Omega e_1)) \\
\dot{P}_1 &= -P_1^2\Omega^2\phi \\
\dot{\hat{k}}_{\mathrm{f}} &= \gamma_2(P_2 + p_2)(\dot{x}_w - \dot{x}_c)e_1 \\
\dot{P}_2 &= -P_2^2(\dot{x}_w - \dot{x}_c)^2
\end{aligned}$$

where small constants (p_1, p_2) and adaptation gains (γ_1 and γ_2) are also added. These additions are not necessary in actual applications and are added here only for better interpretation of the results. For example, because of the unmodified update law almost always converges in actual implementations, initial conditions of the update law may be set up to get good estimates. This is no longer possible. Time trajectories of the estimates are shown in Figure 7. Note also that, although not used in the control law, the estimate of k_{f} helps to improve the accuracy of e_1, and hence the accuracy of the estimate of $\hat{\theta}_1$. Nevertheless, it is possible to include the friction in the model when designing the control law, but the controller will need $\ddot{x}_w - \ddot{x}_c$ signal which is usually quite noisy.

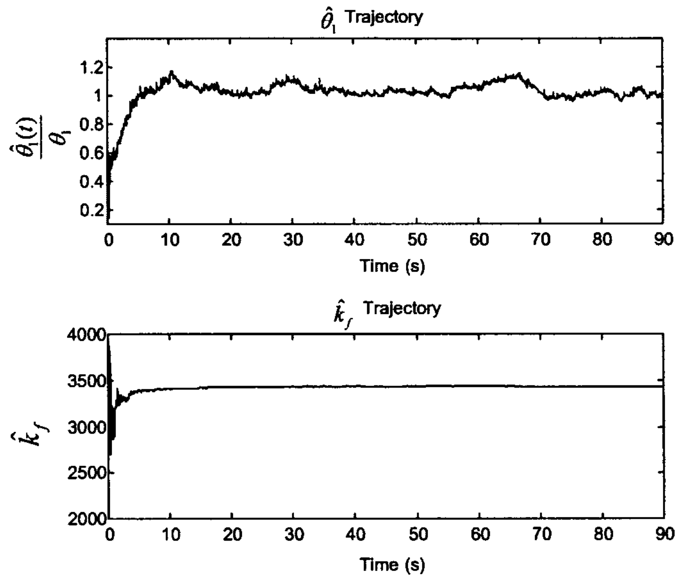

Figure 7. Experimental result of LQ-MARC ($\hat{\theta}_1(t_0) = 0.5\theta_1$).

4. SUMMARY AND CONCLUSIONS

In this paper, a modular adaptive robust control technique was developed. The modular design has all of the original ARC properties with an estimation-based update law instead of a Lyapunov-based update law. In the modular ARC design, the controller is divided into two modules: a control module and an identification module. The system's output tracking accuracy is guaranteed by using *a priori* knowledge of the bound of the unknown parameters and the bound of the generalized uncertainties as well as an arbitrarily set bound on the time derivatives of the estimates. This estimation speed bounds can be guaranteed independently of the control law by using the proposed modifications which also help to guarantee boundedness of the identifier, especially when there are general uncertainties. Although it may yet to be proven about the exact advantage and disadvantage of the proposed modular design compared the non-modular one, the simulations and experimental results show that proposed controller can be beneficial for active suspension applications.

APPENDIX A: SMOOTH PROJECTION MAPPING

Let $\hat{\theta}_i$ denote and estimate of $\theta_i, \pi_i(\ )$ be the smooth projection function, and $\hat{\theta}_{i\pi} = \pi_i(\hat{\theta}_i)$ denote the projected $\hat{\theta}_i$. The smooth projection function $\pi_i(\ )$ is assumed to satisfy the following properties: (1) $\forall \hat{\theta}_i \in \Omega_{\theta_i} \triangleq \{\theta_i : \theta_{im} \leqslant \theta_i \leqslant \theta_{iM}\}, \pi_i(\hat{\theta}_i) = \hat{\theta}_i$; (2) $\forall \hat{\theta}_i, \pi_i(\hat{\theta}_i) \in \Omega_{\hat{\theta}_{i\pi}} \triangleq \{\theta_i : \theta_{im} - \varepsilon_{\pi i} \leqslant \theta_i \leqslant \theta_{iM} + \varepsilon_{\pi i}\}$ where $\varepsilon_{\pi i}$ is a known positive constant; (3) $\pi_i(\hat{\theta}_i)$ is a monotonic function; (4) The derivatives of $\pi_i(\hat{\theta}_i)$ exist and are bounded up to a sufficiently high order. Properties 1–3 are illustrated in Figure A1. In short, $\pi_i(\ )$ maps $\hat{\theta}_i$ into a known bounded region smoothly while it

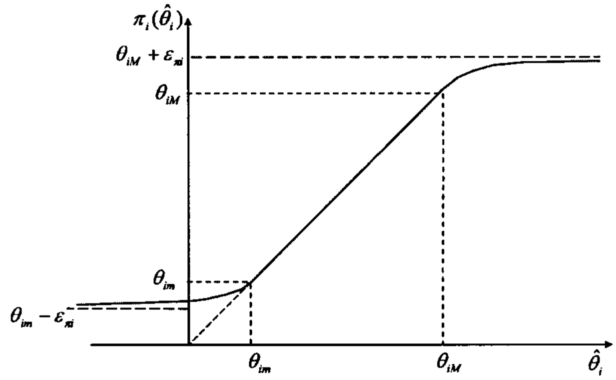

Figure A1. Smooth projection mapping.

acts as an identity map when the estimate lies between θ_{im} and θ_{iM}. Therefore, $\tilde{\theta}_{i\pi} \triangleq \theta_i - \hat{\theta}_{i\pi}$ is bounded, specifically, $|\tilde{\theta}_{i\pi}| \leqslant \pi_i(\infty) - \pi_i(-\infty) \leqslant \theta_{iM} - \theta_{im} + 2\varepsilon_{\pi i}$.

An example of the smooth projection function can be constructed by combining three piecewise continuous functions as follows:

$$\pi_i(\hat{\theta}_i) = \begin{cases} \theta_{im} - F(-\hat{\theta}_i + \theta_{im}), & \hat{\theta}_i < \theta_{im} \\ \hat{\theta}_i, & \theta_{im} \leqslant \hat{\theta}_i \leqslant \theta_{iM} \\ \theta_{iM} + F(\hat{\theta}_i - \theta_{iM}), & \hat{\theta}_i > \theta_{iM} \end{cases}$$

Let $F(x) = \varepsilon_{\pi i} - \Psi(x), \forall x > 0$, where $\Psi(x) = \int_0^x e^{-at^n}\,dt$. The constant a is calculated from $a = (\Gamma(1/n)/n\varepsilon_{\pi i})^n$ where $\Gamma(x)$ is the Euler gamma function. To see that the projection will be continuous and non-decreasing, note that $F'(x) = e^{-ax^n}$. Clearly, $F'(0) = 1, F''(0) = F'''(0) = \cdots = F^{(n)}(0) = 0$, and that $F' > 0,\ \forall x > 0$.

APPENDIX B: PROPOSITION 1: DOMINATING CONTROL SCHEME

Consider a function in the form

$$f(u, x) = F(x)\{A(x) + u\} \tag{B1}$$

where $F(x)$ is a known function of x, $A(x)$ is unknown, and u is the input signal to be designed. Assuming the uncertainty $A(x)$ is dominated by a known function $A_M(x)$ such that

$$|A(x)| \leqslant |A_M(x)|, \quad \forall x$$

then for any $\varepsilon > 0$, a control signal

$$u(x) = -F(x)\frac{1}{4\varepsilon}A_M(x)^2 \tag{B2}$$

yields

$$f(u(x), x) \leqslant \varepsilon, \quad \forall x$$

Furthermore, $F(x)u(x) \leqslant 0$.

Proof
Substitute Equation (B2) into Equation (B1), and completing the square, we have

$$f(x) = -\frac{A_M(x)^2}{4\varepsilon}\left\{\left(F(x) - 2\varepsilon\frac{A(x)}{A_M(x)^2}\right)^2 - 4\varepsilon^2\frac{A(x)^2}{A_M(x)^4}\right\} \leqslant \varepsilon\frac{A(x)^2}{A_M(x)^2} \leqslant \varepsilon \quad \square$$

APPENDIX C: PROOF OF LEMMA 1

The backstepping is the key technique of this proof. By applying the virtual control (α_j) shown in Equation (6) successively, the results can be obtained as follows.

Step 1:

$$\text{Let } z_1 = x_1 - y_r \quad \text{and} \quad V_1 = \tfrac{1}{2}z_1^2 \tag{C1}$$

Differentiating Equation (C1) and substituting Equation (1) and letting $z_2 = x_2 - \alpha_1$, we obtain

$$\dot{V}_1 = z_1 z_2 + z_1(\alpha_1 + \varphi_1(x_1)^{\mathrm{T}}\theta + \Delta_1(t) - \dot{y}_r) \tag{C2}$$

According to Equation (6), the virtual control signal for x_2 is

$$\alpha_1 = -c_1 z_1 + \dot{y}_r - w_1^{\mathrm{T}}\hat{\theta}_\pi - s_1 z_1 \tag{C3}$$

where $w_1^{\mathrm{T}} = \varphi_1^{\mathrm{T}}$ and $s_1 = \frac{1}{4}\{\varphi_1^{2\mathrm{T}}\Pi^2/\varepsilon_{1,\theta} + \Delta_{1M}^2/\varepsilon_{1,1}\}$. Using Equations (C2) and (C3), we have

$$\dot{V}_1 = z_1 z_2 - c_1 z_1^2 + z_1(-s_1 z_1 + \varphi_1^{\mathrm{T}}\tilde{\theta}_\pi + \Delta_1)$$

From Proposition 1 and the facts that $\tilde{\theta}_{k\pi}^2 \leqslant \Pi_k^2$ and $\Delta_k^2 \leqslant \Delta_{kM}^2$, the above equation can be written as

$$\dot{V}_1 \leqslant z_1 z_2 - c_1 z_1^2 + \varepsilon_{1,\theta}\sum_{k=1}^{p}\frac{\tilde{\theta}_{k\pi}^2}{\Pi_k^2} + \varepsilon_{1,1}\frac{\Delta_1^2}{\Delta_{1M}^2} \leqslant \delta_1 z_1 z_2 - \delta_1 c_1 z_1^2 + \varepsilon_1$$

where $\varepsilon_{1\rho} = \varepsilon_{1,\theta} + \varepsilon_{1,1}$. Using Equations (C3) and (1), the dynamic equation for z_1 is

$$\dot{z}_1 = -c_1 z_1 - s_1 z_1 + z_2 + w_1^{\mathrm{T}}\tilde{\theta}_\pi + \Delta_1$$

Step j: Let $z_j = x_j - \alpha_{j-1}$ and $V_j = V_{j-1} + \frac{1}{2}z_j^2$. Differentiating z_j, we have

$$\dot{z}_j = z_{j+1} + \alpha_j + \varphi_j^{\mathrm{T}}\theta + \Delta_j - \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}}{\partial x_k}(x_{k+1} + \varphi_k^{\mathrm{T}}\theta + \Delta_k) - \frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} - \frac{\partial\alpha_{j-1}}{\partial Y^{(j-1)}}\dot{Y}^{(j-1)} \tag{C4}$$

Differentiating V_j and applying Equation (C4), we have

$$\begin{aligned}\dot{V}_j \leqslant \dot{V}_{j-1} + z_j\dot{z}_j \leqslant -\sum_{k=1}^{j-1}(c_k z_k^2 + \varepsilon_k) + z_{j-1}z_j + z_j\Bigg\{& z_{j+1} + \alpha_j + w_j^{\mathrm{T}}\theta + \Delta_j \\ &- \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}}{\partial x_k}(x_{k+1} + \Delta_k) - \frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} - \frac{\partial\alpha_{j-1}}{\partial Y^{(j-1)}}\dot{Y}^{(j-1)}\Bigg\}\end{aligned} \tag{C5}$$

where

$$w_j = \varphi_j - \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_k} \varphi_k$$

From Equation (6), the virtual control α_j is

$$\alpha_j = -c_j z_j - z_{j-1} + \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{j-1}}{\partial Y^{(j-1)}} \dot{Y}^{(j-1)} - w_j^{\mathrm{T}} \hat{\theta}_\pi - s_j z_j \tag{C6}$$

where

$$s_j = -\frac{1}{4}\left\{ \frac{w_j^{2\mathrm{T}} \Pi^2}{\varepsilon_{j,\theta}} + \left(\frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\right)^2 \frac{M^2}{\varepsilon_{j,\dot{\hat{\theta}}}} + \sum_{k=1}^{j-1} \left(\frac{\partial \alpha_{j-1}}{\partial x_k}\right)^2 \frac{\Delta_{kM}^2}{\varepsilon_{j,k}} + \frac{\Delta_{jM}^2}{\varepsilon_{j,j}} \right\}$$

Substitute Equations (C6) into Equation (C5), we have

$$\begin{aligned} \dot{V}_j &\leqslant -\sum_{k=1}^{j-1} (c_k z_k^2 + \varepsilon_k) + z_j z_{j+1} - c_j z_j^2 + \varepsilon_{j,\theta} \sum_{k=1}^{p} \frac{\tilde{\theta}_{k\pi}^2}{\Pi_k^2} + \varepsilon_{j,\dot{\hat{\theta}}} \sum_{k=1}^{p} \frac{\dot{\hat{\theta}}_k^2}{M_k^2} + \sum_{k=1}^{j} \varepsilon_{j,k} \frac{\Delta_k^2}{\Delta_{kM}^2} \\ &\leqslant -\sum_{k=1}^{j} (c_k z_k^2 + \varepsilon_k) + z_j z_{j+1} \end{aligned}$$

where $\varepsilon_j = p\varepsilon_{j,\theta} + p\varepsilon_{j,\dot{\hat{\theta}}} + \sum_{k=1}^{j} \varepsilon_{j,k}$. Finally, from Equation (C4), the dynamic model of z_j is

$$\dot{z}_j = -z_{j-1} - c_j z_j - s_j z_j + z_{j+1} + w_j^{\mathrm{T}} \tilde{\theta}_\pi - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \Delta_j - \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_k} \Delta_k$$

Step n: For step n, the above procedure still applies. However, the virtual control will be used as the control law; i.e. $u = \alpha_n$ and $z_{n+1} \equiv 0$. Given $V_n = V_{n-1} + \frac{1}{2} z_n^2$, it can be shown that $\dot{V}_n \leqslant -\sum_{k=1}^{n} (c_k z_k^2 + \varepsilon_k)$ and

$$\dot{V}_n \leqslant -k_v V_n + k_\varepsilon \tag{C7}$$

where $k_v = 2\min_k (c_k)$ and $k_\varepsilon = \sum_{k=1}^{n} \varepsilon_k$. Note that both k_v and k_ε are positive constants. Since V_n is positive, it is clear from Equation (C7) that V_n converge to the set $\{V_n : V_n \leqslant k_\varepsilon / k_v\}$ with an exponential rate larger than or equal to k_v. As a result, the transient tracking accuracy and steady-state tracking accuracy can be tuned by adjusting k_v and k_ε / k_v, respectively (however, they also affect $V_n(0)$, see [4] for details about a technique to keep $V_n(0)$ constant). Furthermore, from Equation (C7), z_i are bounded. As a result, all states of the system are bounded. To obtain Equation (8), we simply combine the z_j dynamics from each step. □

APPENDIX D: PROOF OF LEMMA 3

Owing to space limitation, we will consider only the case of a least-square update law. This proof can be applied for the gradient update law with minor modifications. To prove (1) and (2), let

$$V = \frac{1}{2} \tilde{\theta}^{\mathrm{T}} \Gamma^{-1} \tilde{\theta} + \frac{\eta}{c_0} \sum_{k=1}^{n} \frac{1}{2} \tilde{e}_k^2 \tag{D1}$$

where $\eta > 1$ and $c_0 = \min_i (c_i)$. Differentiating Equation (D1) and apply Equations (16), (19) and (15), we have

$$\dot{V} \leqslant -\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\hat{\theta}} - \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\Gamma}\Gamma^{-1}\tilde{\theta} - \eta\tilde{e}^{\mathrm{T}}\tilde{e}$$

$$\leqslant -\tilde{\theta}^{\mathrm{T}}\mu(\hat{\theta}) - \tilde{\theta}^{\mathrm{T}}\Theta(\Omega e) + \frac{\phi}{2}\tilde{\theta}^{\mathrm{T}}\Omega\Omega^{\mathrm{T}}\tilde{\theta} - \eta\tilde{e}^{\mathrm{T}}\tilde{e}$$

From Fact 3 and Equation (14), we obtain

$$\dot{V} \leqslant -\mu(\hat{\theta})^{\mathrm{T}}\mu(\hat{\theta}) - \tilde{\theta}^{\mathrm{T}}\Theta(\Omega e) + \frac{\phi}{2}(e-\tilde{e})^{\mathrm{T}}(e-\tilde{e}) - \eta\tilde{e}^{\mathrm{T}}\tilde{e} \tag{D2}$$

There are two cases to be considered.

Case 1: When $|\Omega e| \leqslant M'$ (where M' is the bound of the magnitude of $\Theta(\,)$), Equation (D2) becomes

$$\dot{V} \leqslant -\mu(\hat{\theta})^{\mathrm{T}}\mu(\hat{\theta}) - \tilde{\theta}^{\mathrm{T}}\Omega e + \tfrac{1}{2}e^{\mathrm{T}}e - e^{\mathrm{T}}\tilde{e} - \left(\eta - \tfrac{1}{2}\right)\tilde{e}^{\mathrm{T}}\tilde{e}$$

since $\phi = 1$ in this case. Using Equation (14), we obtain

$$\dot{V} \leqslant -\mu(\hat{\theta})^{\mathrm{T}}\mu(\hat{\theta}) - \left(\eta - \tfrac{1}{2}\right)\tilde{e}^{\mathrm{T}}\tilde{e} - \tfrac{1}{2}e^{\mathrm{T}}e \tag{D3}$$

Case 2: When $|\Omega e| > M'$, we have

$$\dot{V} \leqslant -\mu(\hat{\theta})^{\mathrm{T}}\mu(\hat{\theta}) - \tilde{\theta}^{\mathrm{T}}\Omega e\frac{M'}{|\Omega e|} + \frac{\phi}{2}(e-\tilde{e})^{\mathrm{T}}(e-\tilde{e}) - \eta\tilde{e}^{\mathrm{T}}\tilde{e}$$

Since $\phi = M'/|\Omega e|$ in this case, using Equation (14), we have

$$\dot{V} \leqslant -\mu(\hat{\theta})^{\mathrm{T}}\mu(\hat{\theta}) - (e-\tilde{e})^{\mathrm{T}}e\phi + \frac{\phi}{2}(e-\tilde{e})^{\mathrm{T}}(e-\tilde{e}) - \eta\tilde{e}^{\mathrm{T}}\tilde{e}$$

$$\leqslant -\mu(\hat{\theta})^{\mathrm{T}}\mu(\hat{\theta}) - \frac{\phi}{2}e^{\mathrm{T}}e - \left(\eta - \frac{\phi}{2}\right)\tilde{e}^{\mathrm{T}}\tilde{e}$$

Since $\phi \leqslant 1$, Cases 1 and 2 can be combined to obtain

$$\dot{V} \leqslant -\mu(\hat{\theta})^{\mathrm{T}}\mu(\hat{\theta}) - \frac{\phi}{2}e^{\mathrm{T}}e - \left(\eta - \tfrac{1}{2}\right)\tilde{e}^{\mathrm{T}}\tilde{e} \tag{D4}$$

Since e and Ω are bounded (Lemma 2), there exists a constant m such that $0 < m < \phi/2$. Equation (D4) implies that $\dot{V} \leqslant -\mu(\hat{\theta})^{\mathrm{T}}\mu(\hat{\theta})$ and $\dot{V} \leqslant -me^{\mathrm{T}}e$. These two equations imply that both $\mu(\hat{\theta}) \in L_2$ and $e \in L_2$. To see that, integrate both side of the second equation to obtain

$$\int_0^{\infty} e^{\mathrm{T}}e\,\mathrm{d}t \leqslant \frac{1}{m}\{V(0) - V(\infty)\}$$

which is obtained by dividing the integral into interval not including reset points (t_r) and note that at these points, we have

$$V(t_r^+) - V(t_r) = \tfrac{1}{2}\tilde{\theta}^{\mathrm{T}}(\Gamma^{-1}(t_r^+) - \Gamma^{-1}(t_r))\tilde{\theta}^{\mathrm{T}}$$

Since $\Gamma^{-1}(t_r^+) = I/\rho_0$ and $\Gamma^{-1}(t_r) \geqslant I/\rho_0$, it follows that $V(t_r^+) - V(t_r)$ is negative and can be dropped from the right side of the question. Since V is always positive and is non-increasing, its limit ($V(\infty)$) exists. As a result, we have that $e \in L_2$.

(3) From Equation (16), we have

$$\begin{aligned}\int_0^\infty \dot{\hat{\theta}}^{\mathrm{T}}\dot{\hat{\theta}}\,\mathrm{d}t &\leqslant 2\bar{\lambda}(\Gamma^{\mathrm{T}}\Gamma)\int_0^\infty \mu(\hat{\theta})^{\mathrm{T}}\mu(\hat{\theta}) + \Theta(\Omega e)^{\mathrm{T}}\Theta(\Omega e)\,\mathrm{d}t \\ &\leqslant 2\bar{\lambda}(\Gamma^{\mathrm{T}}\Gamma)\left\{\int_0^\infty \mu(\hat{\theta})^{\mathrm{T}}\mu(\hat{\theta})\,\mathrm{d}t + \int_0^\infty e^{\mathrm{T}}\Omega^{\mathrm{T}}\Omega e\,\mathrm{d}t\right\} \\ &\leqslant 2\bar{\lambda}(\Gamma^{\mathrm{T}}\Gamma)\left\{\int_0^\infty \mu(\hat{\theta})^{\mathrm{T}}\mu(\hat{\theta})\,\mathrm{d}t + \bar{\lambda}(\Omega^{\mathrm{T}}\Omega)\int_0^\infty e^{\mathrm{T}}e\,\mathrm{d}t\right\}\end{aligned}$$

Since $\mu(\hat{\theta}) \in L_2$, $e \in L_2$, and Γ and Ω are bounded, we have $\dot{\hat{\theta}} \in L_2$.

(4) Since $e \in L_2$, and $\tilde{e} \in L_2$, it follows from Equation (14) that $\Omega^{\mathrm{T}}\tilde{\theta} \in L_2$. □

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